Bilan : $\otimes$ et $\oplus \Rightarrow \forall n \in IN; v_n . v_{n+1} = \dfrac{2\pi}{(n+2)(n+3)(n+4)}$.

b) $\forall n \in IN$;

$$v_n . v_{n+1} = \frac{2\pi}{(n+2)(n+3)(n+4)} \Leftrightarrow (v_n)^2 . \frac{v_{n+1}}{v_n} = \frac{2\pi}{(n+2)(n+3)(n+4)}$$

$$\Leftrightarrow (v_n)^2 = \frac{2\pi}{(n+2)(n+3)(n+4)} \times \frac{1}{\frac{v_{n+1}}{v_n}}$$

$$\Leftrightarrow v_n = \sqrt{\frac{2\pi}{(n+2)(n+3)(n+4)} \times \frac{1}{\frac{v_{n+1}}{v_n}}}$$

$$\Leftrightarrow n^{\frac{3}{2}} . v_n = \sqrt{\frac{2\pi n^3}{(n+2)(n+3)(n+4)} \times \frac{1}{\frac{v_{n+1}}{v_n}}}$$

Or $\lim\limits_{n\to+\infty} \dfrac{2\pi n^3}{(n+2)(n+3)(n+4)} = \lim\limits_{n\to+\infty} \dfrac{2\pi n^3}{n^3} = 2\pi$ et $\lim\limits_{n\to+\infty} \dfrac{v_{n+1}}{v_n} = 1$

$$\Rightarrow \lim_{n\to+\infty} \sqrt{\frac{2\pi n^3}{(n+2)(n+3)(n+4)} \times \frac{1}{\frac{v_{n+1}}{v_n}}} = \sqrt{2\pi} \Leftrightarrow \lim_{n\to+\infty} n^{\frac{3}{2}} . v_n = \sqrt{2\pi}.$$

6/ • $v_{2\times 0} = v_0 = \dfrac{2}{3}$ et $\dfrac{2}{(2\times 0+1)(2\times 0+3)} . \dfrac{(2^0 . 0!)^2}{(2\times 0)!} = \dfrac{2}{3}$ car $0! = 1$

$$\Rightarrow v_{2\times 0} = \frac{2}{(2\times 0+1)(2\times 0+3)} . \frac{(2^0 . 0!)^2}{(2\times 0)!}$$

D'où la proposition est vraie pour $p = 0$.

⋆ Soit $q \in IN$. Supposons que $v_{2q} = \dfrac{2}{(2q+1)(2q+3)} . \dfrac{(2^q . q!)^2}{(2q)!}$

Montrons que $v_{2(q+1)} = \dfrac{2}{(2q+3)(2q+5)} . \dfrac{(2^{(q+1)} . (q+1)!)^2}{(2(q+1))!}$

$$v_{2(q+1)} = v_{2q+2} = \frac{2q+2}{2q+5} v_{2q}$$

$$= \frac{2q+2}{2q+5} \times \frac{2}{(2q+1)(2q+3)} . \frac{(2^q . q!)^2}{(2q)!}$$

$$= \frac{2}{(2q+3)(2q+5)} \times \frac{2^2 (q+1)^2}{(2q+2)(2q+1)} . \frac{(2^q . q!)^2}{(2q)!}$$

$$= \frac{2}{(2q+3)(2q+5)} \times . \frac{(2\times 2^q . (q+1) . q!)^2}{(2q+2)!}$$

$$= \frac{2}{(2q+3)(2q+5)} . \frac{(2^{(q+1)} . (q+1)!)^2}{(2(q+1))!}$$

Conclusion : • et ⋆ $\Rightarrow \forall p \in IN; v_{2p} = \dfrac{2}{(2p+1)(2p+3)} . \dfrac{(2^p . p!)^2}{(2p)!}$.

III/1/ $\forall n \in IN^*; e^{u_n} = e^{\text{Log}(n!) - \left(n+\frac{1}{2}\right).\text{Log}n+n} = \dfrac{e^{\text{Log}(n!)}}{e^{\text{Log}\left[n^{n+\frac{1}{2}}\right]}} e^n = \dfrac{n!}{n^{n+\frac{1}{2}}} . e^n$.